SOLUTION OF PROBLEM 155.

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"To find the least distance between two places given by latitude and longitude, taking into account the polar compression."

This question admits two interpretations, viz.:—1, To find the rectilinear distance, that is the chord joining the two points; and 2, To find the shortest distance between the two points on the spheroid or that along the geodesic line.

First Supposition.—Let φ_1, φ_2 be the latitudes and λ_1, λ_2 the longitudes of the two given points respectively then we have their orthogonal coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) as follows:

$$\left.\begin{array}{ll} x_1 = N_1 \cos\varphi_1 \cos\lambda_1, & x_2 = N_2 \cos\varphi_2 \cos\lambda_2, \\ y_1 = N_1 \cos\varphi_1 \cos\lambda_1, & y_2 = N_2 \cos\varphi_2 \cos\lambda_2, \\ z_1 = N_1 (1-e^2) \sin\varphi_1, & z_2 = N_2 (1-e^2) \sin\varphi_2, \end{array}\right\} \tag{1}$$

where $N = \dfrac{a}{\sqrt{(1-e^2\sin^2\varphi)}}$ = normal ending at polar axis, a = equatorial semiaxis and e = eccentricity of meridian. We have then, denoting the straight line connecting the two points by s,

$$\begin{aligned} s &= \sqrt{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]} \\ &= \sqrt{[N_1^2 \cos^2\varphi_1 - 2N_1N_2 \cos\varphi_1 \cos\varphi_2 \cos(\lambda_2-\lambda_1) + N_2^2 \cos^2\varphi_2} \\ &\qquad \overline{+(1-e^2)^2(N_2 \sin\varphi_2 - N_1 \sin\varphi_1)^2]} \\ &= \sqrt{\{ N_1^2 - 2N_1N_2[\cos\varphi_1 \cos\varphi_2 \cos(\lambda_2-\lambda_1) + \sin\varphi_1 \sin\varphi_2]} \\ &\qquad \overline{+N_2^2 - e^2(2-e^2)(N_2 \sin\varphi_2 - N_1 \sin\varphi_1)^2 \}} \end{aligned} \tag{2}$$

If we draw through the center of the spheroid two lines parallel to the normals N_1, N_2 we have, denoting their included angle by σ,

$$\cos\sigma = \sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cos(\lambda_2-\lambda_1). \tag{3}$$

We have then

$$s = \sqrt{[N_1^2 - 2N_1N_2 \cos\sigma + N_2^2 - e^2(2-e^2)(N_2 \sin\varphi_2 - N_1 \sin\varphi_1)^2]}. \tag{4}$$

Join the ends of the normals by a straight line which may be denoted by s', then we have from the rectilinear triangle thus formed

$$s'^2 = N_1^2 - 2N_1N_2 \cos\sigma + N_2^2, \tag{5}$$

hence

$$s = \sqrt{[s'^2 - e^2(2-e^2)(N_2 \sin\varphi_2 - N_1 \sin\varphi_1)^2]}. \tag{6}$$

The geometrical signification of the second term under the radical, which is always a small quantity and may be neglected for two places of nearly the same latitude, is the following:

Let $\Delta z' = N_2 \sin\varphi_2 - N_1 \sin\varphi_1$ = projection of s' on polar axis,
$\Delta z = z_2 - z_1 = (1-e^2)(N_2 \sin\varphi_2 - N_1 \sin\varphi_1)$ = projection of s on polar axis, then $\Delta z'^2 - \Delta z^2 = e^2(2-e^2)(N_2 \sin\varphi_2 - N_1 \sin\varphi_1)^2$ and

$$s = \sqrt{[s'^2 - \Delta z'^2 + \Delta z^2]}, \tag{7}$$

hence

$$s^2 - \Delta z^2 = s'^2 - \Delta z'^2, \tag{8}$$

that is, the projections of s and s' on the equator are equal.

Second Supposition—Any surface of revolution may be represented by the equation

$$z = f(\rho), \tag{1}$$

where z = distance of any point from some plane of reference at right angles to the axis of rotation, which may be called the equator, and ρ = radius of parallel. In this form the equation of the surface is independent of the third coordinate, the longitude λ = inclination of any meridian to the first meridian. In this system of coordinates we have for the element of arc of any curve in space

$$ds = \sqrt{[d\rho^2 + \rho^2 d\lambda^2 + dz^2]}, \tag{2}$$

and the arc of any curve between two fixed points (ρ_1, λ_1, z_1) (ρ_2, λ_2, z_2)

$$s = \int_1^2 \sqrt{[d\rho^2 + \rho^2 d\lambda^2 + dz^2]}. \tag{3}$$

If this arc is to be on the surface $z = f(\rho)$ there will be only two independent variables, viz., ρ and λ in (3); and if we introduce the condition that s shall be the shortest arc on the surface between the given points we must have $\frac{\delta s}{\delta \lambda} = 0$, or else $\frac{\delta s}{\delta \rho} = 0$.

But $\frac{\delta s}{\delta \lambda} = \frac{1}{\delta \lambda} \int_1^2 \frac{\rho^2 d\lambda}{ds} \delta d\lambda = \left[\frac{\rho^2 d\lambda}{ds}\right]_1^2 - \frac{1}{\delta\lambda}\int_1^2 \left(d.\frac{\rho^2 d\lambda}{ds}\right)\delta\lambda = 0.$

The first term vanishes at the limits since the curve is to pass through fixed points, consequently the other term must be 0 also, and therefore

$$d.\frac{\rho^2 d\lambda}{ds} = 0, \text{ or, integrating } \rho^2 d\lambda = c\,ds, \tag{4}$$

and integrating again

$$\int_1^2 \rho^2 d\lambda = cs. \tag{5}$$

This equation shows that the arc of the curve is proportional to the sector of its horizontal projection.

If A is the angle which the geodesic line makes with the meridian or its azimuth, we have evidently

$$\sin A = \rho \frac{d\lambda}{ds} \tag{6}$$

Combining this equation with (4) we have

$$\rho \sin A = c. \tag{7}$$

The constant c is therefore the radius of parallel where the geodesic line meets a meridian at right angles. Combining (4) with (2) we obtain

$$d\lambda = \frac{c}{\rho}\sqrt{\left(\frac{d\rho^2+dz^2}{\rho^2-c^2}\right)}, \tag{8}$$

which is the polar differential equation of the horizontal projection of the geodesic line on any surface of revolution; for it is an equation between the longitude λ, the radius of parallel ρ and an undetermined constant c. To determine this constant we have the condition

$$\lambda_2-\lambda_1 = c\int_1^2 \frac{1}{\rho}\sqrt{\left(\frac{d\rho^2+dz^2}{\rho^2-c^2}\right)}. \tag{9}$$

The length of the arc of the geodesic is then

$$s = \int_1^2 \rho\sqrt{\left(\frac{d\rho^2+dz^2}{\rho^2-c^2}\right)}. \tag{10}$$

For the spheroid we have

$$z = \sqrt{[(1-e^2)(a^2-\rho^2)]}; \tag{11}$$

$$\therefore \quad \lambda_2-\lambda_1 = c\int_1^2 \frac{d\rho}{\rho}\sqrt{\left[\frac{a^2-e^2\rho^2}{(a^2-\rho^2)(\rho^2-c^2)}\right]} \tag{9'}$$

$$s = \int_1^2 \rho d\rho\sqrt{\left[\frac{a^2-e^2\rho^2}{(a^2-\rho^2)(\rho^2-c^2)}\right]}. \tag{10'}$$

The latitudes and longitudes of the two points being the data, we have for the limits of these integrals

$$\rho_1 = a(\cos\varphi_1)\div\varDelta\varphi_1,\ \rho_2 = a(\cos\varphi_2)\div\varDelta\varphi_2, \tag{12}$$

if $\varDelta\varphi = \sqrt{(1-e^2\sin^2\varphi)}$.

Placing $\rho = \sqrt{[c^2+(a^2-c^2)\sin^2\psi]}$ we have

$$\lambda_2-\lambda_1 = c\int_1^2 d\psi\frac{\sqrt{[a^2-c^2e^2-(a^2-c^2)e^2\sin^2\psi]}}{c^2+(a^2-c^2)\sin^2\psi} = \frac{a^2}{\sqrt{(a^2-c^2e^2)}}\,\Pi_1^2\left[\psi,\ e\sqrt{\left(\frac{a^2-c^2}{a^2-c^2e^2}\right)},\ \frac{a^2-c^2}{c^2}\right] - \frac{ce^2}{\sqrt{(a^2-c^2e^2)}}\,F_1^2\left[\psi,\ e\sqrt{\left(\frac{a^2-c^2}{a^2-c^2e^2}\right)}\right] \tag{9''}$$

$$s = \int_1^2 d\psi\sqrt{[a^2-c^2e^2-(a^2-c^2)e^2\sin^2\psi]} = \sqrt{(a^2-c^2e^2)}E_1^2\left[\psi,\ e\sqrt{\left(\frac{a^2-c^2}{a^2-c^2e^2}\right)}\right] \quad \ldots \tag{10''}$$

These elliptics give the complete solution of the problem in theory. The great difficulty consists in the determination of the constant c. If the azimuth of the geodesic were identical with the azimuth as measured by a theodolite and which is the inclination of the vertical circle, passing through the other point, to the meridian;* then equation (7) would give the constant c

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*This field azimuth is given in terms of the latitndes and longitudes of the two points by the formula: $\cot A_1 = \cot \alpha_1 - \frac{e^2\cos\phi_1}{\sin(\lambda_2-\lambda_1)}\cdot\frac{\triangle\phi_2}{\cos\phi_2}\left\{\frac{\sin\phi_2}{\triangle\phi_2}-\frac{\sin\phi_1}{\triangle\phi_1}\right\}$, where $\cot\alpha_1 = [\tan\phi_1 \times\cos\phi_1 - \sin\phi_1\cos(\lambda_2-\lambda_1)]\div\sin(\lambda_2-\lambda_1)$. It may differ from the azimuth of the geodesic line by any amount between 0° and 90°. For instance, if both points are on the equator and nearly 180° apart the geodesic line passes near the pole, its azimuth is therefore nearly 180° while the field azimuth (of course impr'le) would follow the plane of the equator and=270°.

without any difficulty and equation (9) would be superfluous. If the two points are not too distant a value for c from (7) may be used as a first approximation and its correction determined by (9). The constant c being thus found, the length of the arc is found by the evaluation of an elliptic of the second species.

NOTE.—As it is contended that in the published answer to Prof. Hall's Query (see p. 94) the series which represents the value of u converges so slowly that the method is inconvenient, another answer is here submitted as given by CHAS. H. KUMMELL.

To find the most convenient way of computing the numerical value of the definite integral

$$I = \int_0^{\frac{\pi}{2}} d\varphi \sqrt{(\sin\varphi)}. \tag{1}$$

We have

$$\int_0^{\frac{\pi}{2}} d\varphi (\sin\varphi)^{2m-1}(\cos\varphi)^{2n-1} = \tfrac{1}{2}B(m, n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}. \tag{2}$$

Placing $m = \frac{3}{4}$ and $n = \frac{1}{2}$ we have

$$I = \int_0^{\frac{\pi}{2}} d\varphi \sqrt{(\sin\varphi)} = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{4})}. \tag{3}$$

But $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(\frac{3}{4}) = \dfrac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}$ by the theorem: $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$,

$\Gamma(\frac{5}{4}) = \frac{1}{4}\Gamma(\frac{1}{4})$ " " " : $\Gamma(n+1) = n\Gamma(n)$;

therefore

$$I = \frac{(2\pi)^{\frac{3}{2}}}{\Gamma(\frac{1}{4})^2}. \tag{4}$$

The circumference of the lemniscate of Bernoulli may be expressed in terms of $\Gamma(\frac{1}{4})$ as follows:

The polar equation of the lemniscate is $r^2 = a^2\cos 2\theta$. Denoting the circumference by P we have

$$\tfrac{1}{4}P = a\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{(\cos 2\theta)}}. \tag{5}$$

Placing $\theta = \frac{1}{4}\pi - \frac{1}{2}\varphi$ we have

$$\tfrac{1}{4}P = \frac{a}{2}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{(\sin\varphi)}} = \tfrac{1}{4}aB(\tfrac{1}{4}, \tfrac{1}{2}) = \frac{a}{4}\cdot\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} \text{ [by (2)], or}$$

$$P = a\frac{\Gamma(\frac{1}{4})^2}{\sqrt{(2\pi)}}. \tag{6}$$

Combining this with (4) we obtain

$$I = \frac{2a\pi}{P}. \tag{7}$$